TOPIC ①

PS412 Homework 2-2

2.13 At the HERA Collider, 27.5 GeV electrons were collided head-on with 820 GeV protons. Find $\sqrt{s}=?$

$$p_e^\mu=(E_e,\vec{p}_e) \qquad p_p^\mu=(E_p,\vec{p}_p)$$

$$P^\mu_{Total}=P_e^\mu+P_p^\mu=(E_e+E_p,\ \vec{p}_e+\vec{p}_p)$$

$$s=P^\mu_{Total}P_{\mu,Total}=E_e^2+E_p^2+2E_eE_p-p_e^2-p_p^2-2\underbrace{\vec{p}_e\cdot\vec{p}_p}^{\cos 180^\circ=-1}$$

$$=m_e^2+m_p^2+2(E_eE_p+p_ep_p) \qquad p_e\approx E_e,\quad p_p\approx E_p$$

$$s=m_e^2+m_p^2+4E_eE_p$$

At these energies $s\cong 4E_eE_p \qquad \sqrt{s}=2\sqrt{(820)(27.5)}$

$$\boxed{\sqrt{s}\cong 300.3\text{ GeV}}$$

2.14 Compton Scattering

Before:

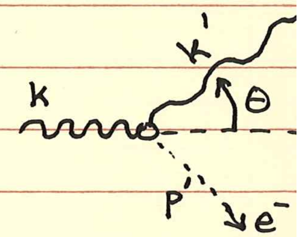
incoming $k=(\overset{E}{k},\vec{k})$

stationary e^- $p=(m_e,\vec{0})$

After:

outgoing $k'=(\overset{E'}{k'},\vec{k}')$

outgoing electron $p'=(E_e',\vec{p}_e')$



4-vectors $\Rightarrow k+p=k'+p' \qquad p'=(k-k')+p$

$$p'^2=(k-k')^2+p^2+2p(k-k')$$

$$p'^2=\underset{=0}{k^2}+\underset{=0}{k'^2}-2kk'+\underset{m_e^2}{p^2}+\underbrace{2p(k-k')}_{2m_e(E-E')}$$

$$p'^2=2m_e(E-E')-2(EE'-EE'\cos\theta)+m_e^2$$

However: $p'^2=E_e'^2-\vec{p}_e'^2=m_e^2$

$$0=\cancel{2}m_e(E-E')-\cancel{2}EE'(1-\cos\theta)$$

$$m_eE'+EE'(1-\cos\theta)=m_eE \Rightarrow E'(m_e+E(1-\cos\theta))=m_eE$$

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2.14 Cont'd

$$E' = \frac{m_e E}{(m_e + E(1-\cos\theta))} \qquad \boxed{E' = \frac{E}{1+\frac{E}{m_e}(1-\cos\theta)}}$$

2.15 Prove $[L_x, L_y] = i\hbar L_z = iL_z$ in natural units using the commutation rules for position and momentum

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ \hat{p}_x & \hat{p}_y & \hat{p}_z \end{vmatrix} = \overbrace{(yp_z - zp_y)}^{L_x}\hat{i} + \overbrace{(zp_x - xp_z)}^{L_y}\hat{j} + \overbrace{(xp_y - yp_x)}^{L_z}\hat{k}$$

$$[L_x, L_y] = (yp_z - zp_y)(zp_x - xp_z) - (zp_x - xp_z)(yp_z - zp_y)$$

$$= yp_z zp_x + zp_y xp_z - \cancel{zp_y zp_x} - \cancel{yp_z xp_z}$$
$$+ \cancel{zp_x zp_y} + \cancel{xp_z yp_z} - zp_x yp_z - xp_z zp_y$$

$$= -yp_x\underbrace{[z, p_z]}_{i\hbar = i} + xp_y\underbrace{[z, p_z]}_{i\hbar = i} = i(xp_y - yp_x) = \boxed{iL_z}$$

b.) Using the commutation relations for the components of angular momentum, prove: $[L^2, L_x] = 0$

$$[L_x^2 + L_y^2 + L_z^2, L_x] = \cancel{[L_x^2, L_x]}^{=0} + [L_y^2, L_x] + [L_z^2, L_x]$$

$\Rightarrow$ Use $[AB, C] = A[B, C] + [A, C]B$

$$[L^2, L_x] = L_y\underbrace{[L_y, L_x]}_{-iL_z} + \underbrace{[L_y, L_x]}_{-iL_z}L_y + L_z\underbrace{[L_z, L_x]}_{iL_y} + \underbrace{[L_z, L_x]}_{iL_y}L_z$$

$$= -iL_yL_z - iL_zL_y + iL_zL_y + iL_yL_z \quad \boxed{= 0}$$

c.) Prove: $L^2 = L_-L_+ + L_z + L_z^2$

$$L_-L_+ = (L_x - iL_y)(L_x + iL_y) = \underbrace{L_x^2 + L_y^2}_{L^2 - L_z^2} + i\underbrace{[L_x, L_y]}_{iL_z}$$

$$L_-L_+ = L^2 - L_z^2 - L_z \Rightarrow \boxed{L^2 = L_-L_+ + L_z^2 + L_z}$$

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TOPIC ③

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2.16 Show that the operators $\hat{S}_i = \frac{1}{2}\sigma_i$, where σ_i are the three Pauli spin matrices:

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Satisfy the same algebra as the angular momentum operators:

$$[S_x, S_y] = iS_z \qquad [S_y, S_z] = iS_x \quad \text{and} \quad [S_z, S_x] = iS_y$$

$$[S_x, S_y] = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{1}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \frac{i}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[S_x, S_y] = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{i}{2}\sigma_z = \boxed{iS_z} \quad \text{because } S_z = \frac{1}{2}\sigma_z$$

... and so on, ... for $[S_y, S_z] = iS_x$ and $[S_z, S_x] = iS$

<u>N.B.</u> Pauli Spin Matrices obey: $\boxed{[\sigma_x, \sigma_y] = i\sigma_z}$, ... etc

b.) Find the eigenvalue(s) of the operator $S^2 = S_x^2 + S_y^2 + S_z^2$ and deduce that the eigenstates of S_z are a suitable representation of a spin ½ particle

$$S^2 = \frac{1}{4}\sigma_x^2 + \frac{1}{4}\sigma_y^2 + \frac{1}{4}\sigma_z^2$$

However, $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \underbrace{I}_{\text{2x2 Identity Matrix}}$

So, $\boxed{S^2 = 3/4\, I}$

Since the spin and angular momentum commutation relations are the same, they must have the same basis functions (i.e., the same quantum state descriptions. In other words

$$L^2 |l\, m_l\rangle = l(l+1)\,|l\, m_l\rangle \qquad L_z|l\, m_l\rangle = m_l\,|l\, m_l\rangle$$

$$S^2 |s\, m_s\rangle = s(s+1)\,|s\, m_s\rangle \qquad S_z|s\, m_s\rangle = m_s\,|s\, m_s\rangle$$

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2.16 cont'd

For a spin ½ particle, a suitable representation would be
spin-up $\Rightarrow |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$
spin-down $\Rightarrow |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$$S^2 |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \underbrace{\tfrac{3}{4}}_{\text{eigenvalue}}|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

and $$S_z |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \underbrace{\tfrac{1}{2}}_{\text{eigenvalue}}|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$